

METRIC PROPERTIES AND DISTORTION IN SOME NILPOTENT GROUPS OF MATRICES

JOSÉ BURILLO AND ERIC LÓPEZ PLATÓN

ABSTRACT. Metric estimates are quantities that approximate the word metric of a finitely presented group up to multiplicative constants. In this paper, they are computed for some nilpotent groups and used to compute the distortion functions of several embeddings between them.

INTRODUCTION

Since the original works by Gromov and other authors in the 1980s, the study of infinite groups from their geometric point of view has experienced a great deal of development. The main tool to study groups as geometric objects is the Cayley graph of the group with a given set of generators, which can be given structure of metric space and whose metric properties are not yet completely well understood, and which give insights on the algebraic properties of the groups.

One of the concepts developed to study groups from the metric point of view is the concept of distortion of a subgroup in a group. This concept, analogous to the geometric concept of distortion of a submanifold, measures the difference between the two metric structures of a finitely generated subgroup inside a group. Namely, its own metric given by its generating set, which compares to the metric induced by the metric of the larger group. This gives rise to the concept of distortion function, which measures the difference between the two metrics. A subgroup is nondistorted if the two metrics are comparable (the distortion function is linear), a concept analogous to that of totally geodesic submanifolds of riemannian manifolds.

The concept of distortion appears already in Gromov's paper [6], and has been studied by several authors, such as Bridson [2], where it is shown that n^r is the distortion function for a pair $G \subset H$ for any rational number $r > 1$, or Sapir and Ol'shanskii (see [7] and [8]), where they give a description of which functions can be obtained as distortion functions of cyclic subgroups in finitely presented groups.

In this paper we study distortion functions obtained by several nilpotent groups embedded into each other. The two families studied are Heisenberg groups and the groups of unipotent upper-triangular matrices. As it could be expected, distortion is polynomial in all cases, and precise degrees are computed for different embeddings between them. The main tool to study this distortion are the estimates of the metric, quantities that can be easily computed for a given element (and its normal form) and which differ from the

The authors are grateful for the support from MEC grant MTM2011-25955, and also from the UPC and the research group COMBGRAF.

actual metric by a multiplicative constant. This method has already been used by the first author for different families, such as Thompson's groups (see [4], [5] or [3]). Hence, these estimates are sufficient to compute the distortion functions, and allow us to obtain precise values for them.

1. BACKGROUND

Recall the definition of distortion function of a subgroup, due to Gromov in [6]:

Definition 1.1. *Let G be a finitely generated group, and $H < G$ a subgroup, also finitely generated. Define the distortion of H in G as:*

$$\Delta_H^G(n) = \max \{ \|x\|_H : x \in H, \|x\|_G \leq n \}$$

As usual, the distortion function depends on the generating set. Two distortion functions for the same subgroup in a group, but with different generating sets (in either the group or the subgroup) differ only by multiplicative constants, and hence only the order of the distortion function is really defined. So we talk about quadratic, polynomial or exponential distortion.

The fact that the distortion is defined only up to multiplicative and additive constants, implies that it is not necessary to know the exact values of $\|x\|$, it is enough to compute them up to constants. This fact gives rise to the following definition:

Definition 1.2. *Given a finitely generated group G , an estimate of the metric or a quasi-metric is a map*

$$E_G : G \longrightarrow \mathbb{N}$$

such that there exist two constants $C, D > 0$ for which

$$\frac{E_G(x)}{C} - D \leq \|x\|_G \leq C E_G(x) + D.$$

for every $x \in G$.

With this definition, if we have quasi-metrics for both G and H , we can redefine the distortion function:

$$\Delta_H^G(n) = \max \{ E_H(x) : x \in H, E_G(x) \leq n \}.$$

2. HEISENBERG GROUPS

The $(2k+1)$ -dimensional Heisenberg group \mathcal{H}_{2k+1} is the group of upper-triangular matrices of the form

$$\begin{pmatrix} 1 & n_1 & n_2 & \cdots & n_k & p \\ & 1 & 0 & \cdots & 0 & m_1 \\ & & 1 & \cdots & 0 & m_2 \\ & & & \ddots & \vdots & \vdots \\ & & & & 1 & m_k \\ & & & & & 1 \end{pmatrix}$$

with integer coefficients, and admits the presentation

$$\langle a_1, \dots, a_k, b_1, \dots, b_k, c \mid [a_i, b_i] = c, [a_i, a_j] = [b_i, b_j] = [a_i, b_j] = [a_i, c] = [b_j, c] = 1 \rangle,$$

where the correspondence between matrices and elements in the presentations is the obvious one: the element a_i corresponds with the matrix which has $n_i = 1$ (and all the other entries equal to zero outside the diagonal). The same way, b_j corresponds to the entry m_j and c to p . It is straightforward to see that any element of \mathcal{H}_{2k+1} admits a unique normal form of the type

$$c^p b_k^{m_k} a_k^{n_k} \dots b_1^{m_1} a_1^{n_1},$$

and this element corresponds to the matrix displayed above.

For the group \mathcal{H}_{2k+1} , we have the following quasi-metric:

Theorem 2.1. *The map*

$$E : \mathcal{H}_{2k+1} \longrightarrow \mathbb{N}$$

defined by

$$E(c^p b_k^{m_k} a_k^{n_k} \dots b_1^{m_1} a_1^{n_1}) = \sum_{i=1}^k |n_i| + \sum_{j=1}^k |m_j| + \sqrt{|p|}.$$

is a quasi-metric for \mathcal{H}_{2k+1} .

Proof. Let $x \in \mathcal{H}_{2k+1}$. We prove first that $\|x\|$ is bounded below by $E(x)$. Let w be a word on the generators which has minimal length $L = \|x\|$. From the word w we rearrange the generators to find the normal form corresponding to x . This process consists of reordering the generators a_i and b_j , and some of the reorderings (when the indices coincide) create new instances of c . But observe that in the rearranging process, we cannot create any generators a_i or b_j . Hence, we have that

$$L \geq |m_i| \quad L \geq |n_j|$$

for $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, k$. For the inequality on p , we have that a generator c is created every time we switch a generator a_i with its match b_i . But clearly, there can be at most L^2 switches of this type. From here, we have that $|p| \leq L^2$, whence the lower bound is deduced.

To see the upper bound for $\|x\|$ we use the normal form itself. Assume $p > 0$ for simplicity, take the inverse if not. Recall that we have that

$$[a_i^n, b_i^n] = c^{n^2}$$

so we will proceed by reducing p to squares. Since every positive integer is sum of at most four squares, we have that there exist four positive integers q_1, q_2, q_3, q_4 such that $p = q_1^2 + q_2^2 + q_3^2 + q_4^2$, and clearly we have that $q_i \leq \sqrt{p}$. Then, we can write c^p as

$$c^p = c^{q_1^2 + q_2^2 + q_3^2 + q_4^2} = [a_1^{q_1}, b_1^{q_1}][a_1^{q_2}, b_1^{q_2}][a_1^{q_3}, b_1^{q_3}][a_1^{q_4}, b_1^{q_4}]$$

and hence we can write our element x as a word of length

$$\sum_{i=1}^k |n_i| + \sum_{j=1}^k |m_j| + 4q_1 + 4q_2 + 4q_3 + 4q_4 \leq \sum_{i=1}^k |n_i| + \sum_{j=1}^k |m_j| + 16\sqrt{p}$$

from which the upper bound is easily deduced. \square

For the special case of \mathcal{H}_3 , the exact value for the word metric with respect to the generators a, b has been computed in [1]. The values obtained there are completely consistent with our results.

From an estimate of the metric like this one, one can always deduce the distortion of cyclic subgroups.

Corollary 2.2. *All cyclic subgroups of \mathcal{H}_{2k+1} are undistorted except those generated by a power of c , which are distorted quadratically.*

The proof is straightforward.

It is clear that if $k \leq l$, we have embeddings of \mathcal{H}_{2k+1} in \mathcal{H}_{2l+1} . For instance, if we take a subset

$$K \subset \{1, 2, \dots, l\}$$

where K has cardinal k , the subgroup of \mathcal{H}_{2l+1} generated by the a_i, b_i for $i \in K$ is a copy of \mathcal{H}_{2k+1} . The corresponding matrices have entries $m_j = n_j = 0$ if $j \notin K$.

Theorem 2.3. *If $k \leq l$, the natural embeddings of \mathcal{H}_{2k+1} into \mathcal{H}_{2l+1} are undistorted.*

Proof. For an element of \mathcal{H}_{2k+1} , its normal forms in \mathcal{H}_{2k+1} and in \mathcal{H}_{2l+1} are identical, so the values of E are the same for \mathcal{H}_{2k+1} and for \mathcal{H}_{2l+1} . \square

3. UPPER-TRIANGULAR MATRICES

A different way of generalizing the group \mathcal{H}_3 is to upper-triangular matrices. Let \mathcal{T}_n be the group of unipotent upper-triangular matrices with integer coefficients. Clearly, $\mathcal{H}_3 = \mathcal{T}_3$. Our goal is to estimate its metric as we have done for \mathcal{H}_{2k+1} , and find the distortion function of the different embeddings.

The standard nilpotent presentation for \mathcal{T}_n is

$$\left\langle \{a_{ij}\}_{1 \leq i < j \leq n} \mid \begin{array}{l} [a_{ij}, a_{kl}] \text{ (for } j \neq k, i \neq l) \\ [a_{ik}, a_{kj}] = a_{ij} \end{array} \right\rangle,$$

where a_{ij} corresponds to the matrix e_{ij} , which is the identity plus an entry equal to 1 in the place (i, j) . Note that since $i < j$, the matrices are all upper-triangular.

From this presentation, a well-known normal form can be deduced, but we will use a slightly modified version, where the order of the generators has been reversed:

Proposition 3.1. *Every element in \mathcal{T}_n can be written uniquely as*

$$\prod_{n \geq j > i \geq 1} a_{ij}^{m_{ij}},$$

with the generators ordered decreasingly as follows: we say that $(i, j) > (k, l)$ if either $j - i > k - l$ or $j - i = k - l$ and $j > l$. Also, the element written in this form corresponds to the matrix which has entry m_{ij} in the (i, j) position.

The proof is elementary and is left to the reader.

The given order, from larger to smaller generators, corresponds with placing first the generators farther from the diagonal, and those in the same diagonal from bottom to top. The reason for this choice is precisely the fact that with this normal form, the exponent m_{ij} appearing in the generator a_{ij} is exactly the (i, j) -th entry of its corresponding matrix. Also, observe that this order and normal form are generalizations of the ones used above in the case of the Heisenberg groups.

Using this normal form, we can also generalize the quasi-metric we have defined for the Heisenberg groups to the upper-triangular groups.

Theorem 3.2. *The map*

$$E : \mathcal{T}_n \longrightarrow \mathbb{N}$$

defined by

$$E \left(\prod_{n \geq j > i \geq 1} a_{ij}^{m_{ij}} \right) = \sum_{1 \leq i < j \leq n} |m_{ij}|^{\frac{1}{j-i}}.$$

is a quasi-metric for \mathcal{T}_n .

The proof is cumbersome because of the complicated notation with subindices and fractional powers, but the idea is quite simple and it is completely analogous to the one used in the Heisenberg group.

Proof. So we proceed as in the case of \mathcal{H}_{2k+1} . To see that $\|x\|$ is bounded from below by $E(x)$, we take a shortest word for x , with length $L = \|x\|$, and proceed to find its normal form from it. We need to estimate the number of extra generators appearing in this process. The bound will follow from the following lemma.

Lemma 3.3. *For $1 \leq i < j \leq n$, there exist constants $C_{ij} > 0$ such that in this process of constructing the normal form, the number of instances of the generator a_{ij} is always bounded above by $C_{ij}L^{j-i}$. In particular, we have that $|m_{ij}| \leq C_{ij}L^{j-i}$.*

Proof of the lemma. The result is best seen by induction on $j - i$. The case $j - i = 1$ follows from the fact that new generators a_{ij} with $j - i = 1$ will never appear in the process. Hence, we have that their number is bounded above by L at all times, and then $|m_{ij}| \leq L$.

Assume now that $j - i > 1$. Initially the number of generators a_{ij} is obviously bounded by L . But each time we switch a generator a_{ik} with a_{kj} , for any k satisfying $i < k < j$, the corresponding relator indicates that a new generator a_{ij} is created. Using the induction hypotheses, we know that along the whole process, the total number of generators a_{ik} is bounded by $C_{ik}L^{k-i}$, and similarly for a_{kj} . According to this, the total number of generators a_{ij} is always bounded by

$$L + \sum_{k=i+1}^{j-1} C_{ik}L^{k-i}C_{kj}L^{j-k} \leq \left(1 + \sum_{k=i+1}^{j-1} C_{ik}C_{kj} \right) L^{j-i}.$$

Taking this constant as C_{ij} we have the desired bound. \square

From here we can deduce the lower bound by taking the appropriate constant, combining all the C_{ij} .

To prove the upper bound for $\|x\|$ in terms of $E(x)$ we proceed as in the Heisenberg case. The normal form has length $\sum |m_{ij}|$, so we need to introduce the corresponding roots. To do that, we note the following commutator equality:

$$[a_{i,i+1}^q, a_{i+1,i+2}^q, \dots, a_{j-1,j}^q] = a_{ij}^{q^{j-i}}$$

where $[x_1, x_2, \dots, x_k]$ is defined recursively for $k \geq 3$ as $[[x_1, x_2, \dots, x_{k-1}], x_k]$. Note also that by switching two generators in a commutator, we obtain the inverse, i.e. with exponent $-q^{j-i}$, so we will not complicate the discussion considering positive and negative powers.

To prove this upper bound, we use again the normal form. If $j - i > 1$, the term $a_{ij}^{m_{ij}}$ will be rewritten as a product of commutators. In the Heisenberg case we used Lagrange's theorem about the sum of four squares, so in this case we need to use its generalization:

Theorem 3.4 (Hilbert-Waring). *For every positive integer k there exists a number $g(k)$ such that any positive integer can be written as a sum of at most $g(k)$ k -th powers.*

So decompose $|m_{ij}|$ as sum of $(j - i)$ -th powers, at most $g(j - i)$ of them. Let q^{j-i} be one of them, and observe that $q \leq |m_{ij}|^{1/(j-i)}$. Rewrite now the power using the commutator:

$$a_{ij}^{q^{j-i}} = [a_{i,i+1}^q, a_{i+1,i+2}^q, \dots, a_{j-1,j}^q].$$

This process replaces a word of length q^{j-i} by one which is linear in q , and hence the piece $a_{ij}^{q^{j-i}}$ is replaced by a word whose length is bounded above by $C|m_{ij}|^{1/(j-i)}$, where C only depends on the commutator, and the number of these words is bounded above by the Hilbert-Waring constant $g(j - i)$. So finally, the whole term $a_{ij}^{m_{ij}}$ is replaced by a word of length at most

$$g(j - i) C |m_{ij}|^{\frac{1}{j-i}}$$

as desired. □

Again we can deduce from this estimate the distortion of a cyclic subgroup:

Corollary 3.5. *A cyclic subgroup of \mathcal{T}_k generated by an element x has distortion n^{j-i} , where a_{ij} is the smallest generator (in the generator order) appearing in the normal form for x .*

And now that all the quasi-metrics are determined, we can find the distortion functions for the embeddings among all the groups \mathcal{H}_{2k+1} and \mathcal{T}_n .

Theorem 3.6. *The distortion function for \mathcal{H}_{2k+1} as a subgroup of \mathcal{T}_{k+2} is polynomial of degree k .*

Proof. Take the standard element of \mathcal{H}_{2k+1}

$$x = \begin{pmatrix} 1 & n_1 & n_2 & \cdots & n_k & p \\ & 1 & 0 & \cdots & 0 & m_1 \\ & & \ddots & \ddots & \vdots & m_2 \\ & & & \ddots & 0 & \vdots \\ & & & & 1 & m_k \\ & & & & & 1 \end{pmatrix}$$

and look at the values of the two quasi-metrics for it:

$$\begin{aligned} E_{\mathcal{H}}(x) &= \sum_{i=1}^k |n_i| + \sum_{j=1}^k |m_j| + \sqrt{|p|} \\ E_{\mathcal{T}}(x) &= |n_1| + |m_1| + \sqrt{|n_2|} + \sqrt{|m_2|} + \cdots + |n_k|^{1/k} + |m_k|^{1/k} + |p|^{1/(k+1)} \end{aligned}$$

We have then that

$$E_{\mathcal{H}} \leq E_{\mathcal{T}}^k$$

because k is at least 1, and then the exponents of $|p|$ satisfy $1/2 \leq k/(k+1)$. Hence, the distortion function is bounded above by a polynomial of degree k . To see that it is exactly a polynomial of this degree, consider the family of elements given by the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & n & 0 \\ & 1 & \cdots & 0 & 0 & 0 \\ & & \ddots & \vdots & \vdots & \vdots \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}$$

whose quasi-metric is n in \mathcal{H}_{2k+1} but $n^{1/k}$ in \mathcal{T}_{k+2} , so it realizes the upper bound. \square

And finally, we can study the distortion of the different embeddings of \mathcal{T}_k inside \mathcal{T}_l for $k < l$.

Theorem 3.7. *Given positive integers $k < l$, for each r such that $1 \leq r \leq l - k + 1$ there is an embedding of \mathcal{T}_k inside \mathcal{T}_l with distortion n^r .*

In particular, observe that inside \mathcal{T}_{k+1} there are subgroups isomorphic to \mathcal{T}_k which are undistorted, and some other subgroups \mathcal{T}_k which are quadratically distorted.

Proof. It is straightforward to see that the embedding of \mathcal{T}_k in \mathcal{T}_l as a block in the upper-left corner is undistorted, because the value of the quasi-metric of an element is the same for both groups.

To see an embedding with distortion n^{l-k+1} , take a k -by- k upper-triangular matrix and split it in non-empty blocks

$$A = \left(\begin{array}{c|c} U_a & M \\ \hline 0 & U_b \end{array} \right)$$

where U_a and U_b are unipotent upper-triangular matrices of sizes $a \times a$ and $b \times b$ respectively. The block M is $a \times b$, and observe $a + b = k$. Embed it as an $l \times l$ matrix

as

$$A' = \left(\begin{array}{c|c|c} U_a & 0 & M \\ \hline 0 & I_{l-k} & 0 \\ \hline 0 & 0 & U_b \end{array} \right).$$

where I_{l-k} is just the identity matrix of size $l - k$. In this embedding, an entry in A in the spot (i, j) , located in the submatrix M , is shifted to the position $(i, j + l - k)$ in the matrix A' . In particular, the worst distortion is given by the lower-left entry in M , say $m_{a,a+1}$, which is in position $(a, a + 1)$ in A , and it shifts to $(a, a + l - k + 1)$ in A' . So the contributions of this entry to the two quasi-metrics are $|m_{a,a+1}|$ in \mathcal{T}_k and $|m_{a,a+1}|^{1/(l-k+1)}$ in \mathcal{T}_l , so it is easily seen that the distortion is n^{l-k+1} .

To find an intermediate distortion n^r with $1 < r < l - k + 1$, embed \mathcal{T}_k with distortion n^r inside \mathcal{T}_{k+r-1} , and then embed this one without distortion inside \mathcal{T}_l . \square

REFERENCES

- [1] Sébastien Blachère. Word distance on the discrete Heisenberg group. *Colloq. Math.*, 95(1):21–36, 2003.
- [2] Martin R. Bridson. Fractional isoperimetric inequalities and subgroup distortion. *J. Amer. Math. Soc.*, 12(4):1103–1118, 1999.
- [3] José Burillo and Sean Cleary. Metric properties of higher-dimensional Thompson’s groups. *Pacific J. Math.*, 248(1):49–62, 2010.
- [4] José Burillo, Sean Cleary, and Melanie Stein. Metrics and embeddings of generalizations of Thompson’s group F . *Trans. Amer. Math. Soc.*, 353(4):1677–1689 (electronic), 2001.
- [5] José Burillo, Sean Cleary, Melanie Stein, and Jennifer Taback. Combinatorial and metric properties of Thompson’s group T . *Trans. Amer. Math. Soc.*, 361(2):631–652, 2009.
- [6] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.
- [7] Alexander Yu. Ol’shanskii. On the distortion of subgroups of finitely presented groups. *Mat. Sb.*, 188(11):51–98, 1997.
- [8] Alexander Yu. Ol’shanskii and Mark V. Sapir. Length and area functions on groups and quasi-isometric Higman embeddings. *Internat. J. Algebra Comput.*, 11(2):137–170, 2001.

DEPARTAMENT DE MATEMÀTICA APLICADA IV, EETAC-UPC, C/ESTEVE TORRADES 5, 08860 CASTELLDEFELS, BARCELONA, SPAIN

E-mail address: burillo@ma4.upc.edu

DEPARTAMENT DE MATEMÀTICA APLICADA IV, UPC, C/JORDI GIRONA 1-3, 08034 BARCELONA, SPAIN

E-mail address: eric.lopez@ma4.upc.edu